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1997 J. Phys. A: Math. Gen. 30 6831

(http://iopscience.iop.org/0305-4470/30/19/021)

J. Phys. A: Math. Gen. **30** (1997) 6831–6840. Printed in the UK PII: S0305-4470(97)84467-8

Physical origin of the Runge–Lenz vector

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Received 27 May 1997

Abstract. The dynamical symmetry of the non-relativistic Kepler problem has attracted much attention in the scientific literature. In the present paper, we show that the Runge–Lenz vector, which accounts for the presence of this symmetry, has its physical origin in the generator of Lorentz transformations for the relativistic *two-body* problem. We reach this conclusion by considering the relativistic two-body problem, for the electromagnetic as well as for the gravitational interaction, in the $1/c^2$ approximation.

1. Introduction

It is well known that the dynamical symmetry of the non-relativistic Kepler problem manifests itself through the so-called Runge–Lenz vector† which is a constant of the motion—both in classical mechanics and in quantum mechanics. In classical mechanics [2, 3], the Runge–Lenz vector confines planetary motion to conic sections fixed in space. In quantum mechanics [4, 5], the corresponding vector operator is responsible for the accidental degeneracy of the energy levels of the hydrogen atom with respect to the angular-momentum quantum number l.

The form of the Runge–Lenz vector associated with a particle moving in the 'Coulomb field' κ/r is the following:

$$\boldsymbol{M} = \frac{\boldsymbol{p} \times \boldsymbol{l}}{\mu} + \frac{\kappa}{r}\boldsymbol{r} \tag{1}$$

where μ is the mass of the particle, $\boldsymbol{p}$ its linear momentum, and $\boldsymbol{l}$ its angular momentum $\boldsymbol{r} \times \boldsymbol{p}$. This expression is the classical one; in quantum mechanics, the vector $\boldsymbol{p} \times \boldsymbol{l}$ is replaced by the Hermitian vector operator $\frac{1}{2}(\boldsymbol{p} \times \boldsymbol{l} - \boldsymbol{l} \times \boldsymbol{p})$.

The following Poisson-bracket relations involving the components of the angular-momentum vector and the Runge–Lenz vector hold:

$$[l_i, l_j] = \sum_k \epsilon_{ijk} l_k \tag{2}$$

$$[l_i, M_j] = \sum_k \epsilon_{ijk} M_k \tag{3}$$

$$[M_i, M_j] = -\frac{2}{\mu} H \sum_k \epsilon_{ijk} l_k. \tag{4}$$

† The history and the 'pre-history' of the Runge–Lenz vector has, in particular, been studied by Goldstein [1]. As a result of his research, Goldstein concludes that it would seem most fitting to refer to the Runge–Lenz vector as the Hermann–Bernoulli–Laplace vector.

0305-4470/97/196831+10$19.50

In these relations, $[f, g]$ is the Poisson bracket between the dynamical functions f and g. H is the non-relativistic Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\kappa}{r} \tag{5}$$

and ϵ_{ijk} is the Levi–Civita symbol. The fact that $\boldsymbol{M}$ is a constant of the motion implies the Poisson-bracket relation

$$[H, \boldsymbol{M}] = \mathbf{0}. \tag{6}$$

The quantum-mechanical expressions that correspond to the relations (4) and (6) are obtained by the substitution

$$[f, g] \rightarrow \frac{1}{\mathrm{i}\hbar}(\hat{f}\hat{g} - \hat{g}\hat{f}) \tag{7}$$

where $\hat{f}\hat{g} - \hat{g}\hat{f}$ is the commutator between the operators $\hat{f}$ and $\hat{g}$†.

For a fixed energy, E, relations (4) may be given a group-theoretical interpretation by rewriting the last of them. For $E < 0$, this is done by defining a vector $\boldsymbol{M}'$ with the components

$$M_i' = \sqrt{-\frac{\mu}{2E}} M_i. \tag{8}$$

We get then

$$[M_i', M_j'] = \sum_k \epsilon_{ijk} l_k. \tag{9}$$

Thus, l_i and M_j' may be considered as generators of the four-dimensional orthogonal group.

For $E > 0$ we define instead

$$M_i'' = \sqrt{\frac{\mu}{2E}} M_i \tag{10}$$

and get

$$[M_i'', M_j''] = -\sum_k \epsilon_{ijk} l_k. \tag{11}$$

Here l_i and M_j'' may be considered as generators of the Lorentz group.

Finally, for $E = 0$, we encounter the generators of the group of rigid motions in the three-dimensional Euclidean space.

The group-theoretical classification of the dynamical symmetry of the Kepler problem was first clarified by Fock [7], Bargmann [8], and Hulthén and Klein [9]. A detailed review is due to Bander and Itzykson [10].

We note that the above symmetry groups are different for $E < 0$, $E > 0$ and $E = 0$. They are elegant mathematical constructs, by means of which the dynamics of the Kepler problem may be described in a most useful and interesting way. However, so far, the presence of the dynamical symmetry has not been tied to a generally accepted physical invariance principle. Nor has it given rise to the discovery of new general invariance principles. The description that has been developed is, accordingly, a purely phenomenological one. Thus, it is relevant and important to ask the question, *Does*

† The relation (7) is generally valid when $[f, g]$ may be written as a linear combination of Poisson brackets between functions of which at least one is no more than quadratic in the components of $\boldsymbol{r}$ or $\boldsymbol{p}$. Otherwise, great care must be exercised in setting up the correspondence between classical and quantum-mechanical relations. See, for instance, [6].

the Runge–Lenz vector have a deeper physical origin? We have been able to derive an affirmative answer to this question.

What we shall show in what follows is, in fact, that it is possible to tie the dynamical symmetry of the non-relativistic Kepler problem to the classical invariance principles of special relativity‡. However, in order to do so, it is essential to realize that the non-relativistic Kepler problem is the zero-order description of a relativistic two-body problem rather than the zero-order description of a one-body problem. The mass μ that appears in the equations above is, accordingly, the reduced mass of a two-body problem. But whereas the introduction of a reduced mass eliminates the centre-of-mass motion in the non-relativistic description—and thus effectively reduces the problem to one of a single particle—this is no longer the case in a proper relativistic description. Thus, it is absolutely necessary to investigate the relativistic *two-body* problem in order to discover the connection between the dynamical symmetry of the non-relativistic Kepler problem and special relativity.

Now, there are two kinds of force that give rise to a potential of the form κ/r, namely the electromagnetic force and the gravitational force. We shall base our discussion on the former, but show at the end that the same conclusions are obtained by invoking the gravitational force. We shall assume that the motion of the particles considered may be described by classical mechanics, and also adopt a description in which retardation effects are taken into account to second-order terms in $1/c$. The restriction to second-order terms is necessary to avoid including energy loss through radiation. Dissipation of energy due to dipole radiation goes as $1/c^3$. A description that goes beyond terms of order $1/c^2$ must, therefore, also include the degrees of freedom of the electromagnetic field. An important exception occurs if the particles have the same charge-to-mass ratio. The dipole radiation is then postponed to order $1/c^5$, and a particle dynamics exists to order $1/c^4$. We shall not consider this special case in the present work.

The $1/c^2$ description was first set up by Darwin [12] in 1920. It is often referred to as post-Newtonian mechanics. The $1/c^4$ order mechanics that may be set up for particles with the same charge to mass ratio is similarly referred to as post-post-Newtonian mechanics [13, 14].

The invariance group of the electromagnetic N-body problem is the inhomogeneous Lorentz group (the Poincaré group). It is well known that this group forces ten constants of the motion upon a physical system; these are the generators of the infinitesimal operations of the group. The total momentum $\boldsymbol{P}$ and the total energy $\mathcal{H}$ generate translations in space and time, and go together to form a four-vector $(\boldsymbol{P}, \mathcal{H}/c)$. With the inhomogeneous Lorentz group described as the semidirect product of a four-dimensional translation group and the homogeneous Lorentz group, $\boldsymbol{P}$ and $\mathcal{H}$ go with the translation group. The generators for the homogeneous Lorentz group are the total angular momentum $\boldsymbol{L}$, generating rotations in three-space, and a polar vector $\boldsymbol{K}$ which generates homogeneous Lorentz transformations without rotations. Together, $\boldsymbol{L}$ and $\boldsymbol{K}$ define an antisymmetric four-tensor $(\boldsymbol{L}, c\boldsymbol{K})$.

The Lagrangian of the electromagnetic N-body problem in the $1/c^2$ approximation (the Darwin Lagrangian) is reproduced in section 2. To make the presentation reasonably self-contained, we also give the expressions for the Hamiltonian and the other constants of the motion. It is the vector $\boldsymbol{K}$ that is of primary interest in the present context.

In section 3, we specialize to the electromagnetic two-body problem and focus on the centre-of-momentum system. By eliminating the centre of mass coordinates, an expression is derived for $\boldsymbol{K}$ in terms of the relative coordinates. It is shown that this expression does,

‡ Some of the conclusions arrived at in the present paper were published in a preliminary form as a letter several years ago [11].

in fact, identify the Runge–Lenz vector. Having thus determined the physical origin of the Runge–Lenz vector, we show that the Poisson-bracket relation (6) is a simple consequence of the fact that $\boldsymbol{K}$ is a constant of the motion.

In section 4, we consider the gravitational two-body problem and show that the Runge–Lenz vector emerges in a similar way in this case. In section 5 we present our conclusions.

2. Post-Newtonian description of an electromagnetic N-body system

The Darwin Lagrangian for a system of N particles, with rest masses $m_1, m_2, \ldots, m_N$ and charges $q_1, q_2, \ldots, q_N$, has the following form ([12], [15, §27], [16, §65]):

$$\mathcal{L} = -\sum_i m_i c^2 + \frac{1}{2}\sum_i m_i v_i^2 + \frac{1}{8c^2}\sum_i m_i v_i^4 - \frac{1}{2}\sum_{i,j}{}' \frac{q_i q_j}{r_{ij}} + \frac{1}{4c^2}\sum_{i,j}{}' q_i q_j \left(\frac{\boldsymbol{v}_i \cdot \boldsymbol{v}_j}{r_{ij}} + \frac{(\boldsymbol{v}_i \cdot \boldsymbol{r}_{ij})(\boldsymbol{v}_j \cdot \boldsymbol{r}_{ij})}{r_{ij}^3} \right) \tag{12}$$

where $\boldsymbol{v}_i$ is the velocity of the ith particle, with position vector $\boldsymbol{r}_i$, and

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j \qquad r_{ij} = |\boldsymbol{r}_{ij}|. \tag{13}$$

A prime on a summation symbol indicates that terms for which two indices become equal are to be omitted in the corresponding double sum.

The momentum associated with $\boldsymbol{r}_i$ is

$$\boldsymbol{p}_i = \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} = \left(\frac{\partial \mathcal{L}}{\partial v_{ix}}, \frac{\partial \mathcal{L}}{\partial v_{iy}}, \frac{\partial \mathcal{L}}{\partial v_{iz}} \right) \tag{14}$$

with v_{ix}, v_{iy} and v_{iz} denoting the Cartesian coordinates of $\boldsymbol{v}_i$. We get

$$\boldsymbol{p}_i = m_i \boldsymbol{v}_i + \frac{m_i v_i^2}{2c^2}\boldsymbol{v}_i + \frac{1}{2c^2}\sum_{j \neq i} q_i q_j \left(\frac{\boldsymbol{v}_j}{r_{ij}} + \frac{\boldsymbol{v}_j \cdot \boldsymbol{r}_{ij}}{r_{ij}^3}\boldsymbol{r}_{ij} \right). \tag{15}$$

The Hamiltonian is constructed by the usual prescription

$$\mathcal{H} = \sum_i \boldsymbol{p}_i \cdot \boldsymbol{v}_i - \mathcal{L} \tag{16}$$

and, to order $1/c^2$, becomes

$$\mathcal{H} = \sum_i m_i c^2 + \sum_i \frac{p_i^2}{2m_i} - \frac{1}{c^2}\sum_i \frac{p_i^4}{8m_i^3} + \frac{1}{2}\sum_{i,j}{}' \frac{q_i q_j}{r_{ij}} - \frac{1}{4c^2}\sum_{i,j}{}' \frac{q_i q_j}{m_i m_j} \left(\frac{\boldsymbol{p}_i \cdot \boldsymbol{p}_j}{r_{ij}} + \frac{(\boldsymbol{p}_i \cdot \boldsymbol{r}_{ij})(\boldsymbol{p}_j \cdot \boldsymbol{r}_{ij})}{r_{ij}^3} \right). \tag{17}$$

For the total momentum of the system we have the usual expression

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i \tag{18}$$

and similarly for the total angular momentum

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i. \tag{19}$$

Following Fock [15, §27] the analytical expression for the vector $\boldsymbol{K}$ may be derived from the variation of the Lagrangian as obtained by subjecting the N-particle system to an infinitesimal Lorentz transformation without rotation, followed by a transformation to simultaneity in the new reference frame. The full transformation is generated by a function of the form $\boldsymbol{K}\cdot\boldsymbol{w}$, with $\boldsymbol{w}$ denoting the relative velocity of the two frames. The expression for $\boldsymbol{K}$ is

$$\boldsymbol{K} = -t\boldsymbol{P} + \sum_i \left[m_i\boldsymbol{r}_i + \frac{1}{2c^2}\left(\frac{p_i^2}{m_i} + \sum_{j\neq i}\frac{q_iq_j}{r_{ij}}\right)\boldsymbol{r}_i\right]. \tag{20}$$

By defining the energy associated with the ith particle to be

$$\mathcal{E}_i = m_ic^2 + \frac{p_i^2}{2m_i} + \frac{1}{2}\sum_{j\neq i}\frac{q_iq_j}{r_{ij}} \tag{21}$$

and introducing the centre of inertia, $\boldsymbol{R}_{\mathrm{CI}}$, by the relation

$$\mathcal{H}\boldsymbol{R}_{\mathrm{CI}} = \sum_i \mathcal{E}_i\boldsymbol{r}_i \tag{22}$$

the expression (20) takes on the form

$$\boldsymbol{K} = \frac{\mathcal{H}}{c^2}\boldsymbol{R}_{\mathrm{CI}} - t\boldsymbol{P}. \tag{23}$$

The fact that $\boldsymbol{K}$ is constant represents, then, the law of motion of the centre of inertia.

The vector $\boldsymbol{K}$ depends explicitly on time, except in the centre-of-momentum system which is the Lorentz frame for which $\boldsymbol{P} = \boldsymbol{0}$. In this particular reference system, $\boldsymbol{K}$ becomes a time-independent constant of the motion.We shall now show that, for the electromagnetic two-body problem, this constant of the motion identifies the Runge–Lenz vector.

3. The electromagnetic two-body problem in the centre-of-momentum system

In the case of only two particles, we shall replace $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by the vectors

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{24}$$

with $\boldsymbol{R}$ being the position vector of the centre of mass and $\boldsymbol{r}$ the position vector for the relative motion. The corresponding momenta are

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 \qquad \boldsymbol{p} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1+m_2}. \tag{25}$$

We also note the inverse relations

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1+m_2}\boldsymbol{r} \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1+m_2}\boldsymbol{r} \tag{26}$$

and

$$\boldsymbol{p}_1 = \boldsymbol{p} + \frac{m_1}{m_1+m_2}\boldsymbol{P} \qquad \boldsymbol{p}_2 = -\boldsymbol{p} + \frac{m_2}{m_1+m_2}\boldsymbol{P}. \tag{27}$$

Exploiting these relations, we obtain the following expression for $\boldsymbol{K}$:

$$\begin{aligned}\boldsymbol{K} = {} & -t\boldsymbol{P} + (m_1+m_2)\boldsymbol{R} + \frac{1}{c^2}\left(\frac{P^2}{2(m_1+m_2)} + \frac{p^2}{2\mu} + \frac{q_1q_2}{r}\right)\boldsymbol{R} \\ & - \frac{1}{2c^2}\frac{m_1-m_2}{m_1+m_2}\left(\frac{p^2}{\mu} + \frac{q_1q_2}{r}\right)\boldsymbol{r} + \frac{1}{c^2}\frac{1}{m_1+m_2}(\boldsymbol{p}\cdot\boldsymbol{P})\boldsymbol{r}\end{aligned} \tag{28}$$

where μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{29}$$

In the centre-of-momentum system, $\boldsymbol{P} = \boldsymbol{0}$, we get

$$\boldsymbol{K} = (m_1 + m_2)\boldsymbol{R} + \frac{1}{c^2}\left(\frac{p^2}{2\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{R} - \frac{1}{2c^2}\frac{m_1 - m_2}{m_1 + m_2}\left(\frac{p^2}{\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{r}. \tag{30}$$

Similarly, the Hamiltonian becomes

$$\mathcal{H} = (m_1 + m_2)c^2 + \frac{p^2}{2\mu} + \frac{q_1 q_2}{r} - \frac{1}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right)p^4 + \frac{q_1 q_2}{2c^2}\frac{1}{m_1 m_2}\left(\frac{p^2}{r} + \frac{(\boldsymbol{p}\cdot\boldsymbol{r})^2}{r^3}\right). \tag{31}$$

To order $1/c^2$ we have

$$\frac{1}{c^2}\mathcal{H} = m_1 + m_2 + \frac{1}{c^2}H \tag{32}$$

where

$$H = \frac{p^2}{2\mu} + \frac{q_1 q_2}{r} \tag{33}$$

is the non-relativistic Hamiltonian. Hence, the expression (30) for $\boldsymbol{K}$ may be written

$$\boldsymbol{K} = \frac{1}{c^2}\mathcal{H}\boldsymbol{R} - \frac{1}{2c^2}\frac{m_1 - m_2}{m_1 + m_2}\left(\frac{p^2}{\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{r}. \tag{34}$$

We also note that equation (19) becomes

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}. \tag{35}$$

In the centre-of-momentum system, $\boldsymbol{P}$ is no longer a dynamical variable. The same holds, therefore, for the corresponding position vector $\boldsymbol{R}$. The expressions (31) and (35) for $\mathcal{H}$ and $\boldsymbol{L}$, respectively, are already independent of $\boldsymbol{R}$, but the expression for $\boldsymbol{K}$ in (34) is not. $\boldsymbol{R}$ must be eliminated from this expression in order for $\boldsymbol{K}$ to be a proper dynamical function. The way to do this is to set up the equation of motion for $\boldsymbol{R}$ with $\boldsymbol{P}$ different from zero, and then solve that equation in the limit $\boldsymbol{P} = \boldsymbol{0}$.

That $\boldsymbol{R}$ is expressible as a function of $\boldsymbol{r}$ and $\boldsymbol{p}$ when $\boldsymbol{P} = \boldsymbol{0}$ was already realized by Darwin [12]. He referred to the centre of mass as the *centroid* and determined its motion in the particular case $q_1 = -q_2$, for a bound quasi-elliptic orbital motion. Here, we shall consider the general case.

With $\mathcal{H}$ being the Hamiltonian (17), Hamilton's equation for $\boldsymbol{R}$ gives

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \left[\frac{\partial\mathcal{H}}{\partial\boldsymbol{P}}\right]_{\boldsymbol{P}=\boldsymbol{0}}. \tag{36}$$

By exploiting the expressions of (27), we may instead write

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \left[\frac{m_1}{m_1 + m_2}\frac{\partial\mathcal{H}}{\partial\boldsymbol{p}_1} + \frac{m_2}{m_1 + m_2}\frac{\partial\mathcal{H}}{\partial\boldsymbol{p}_2}\right]_{\substack{\boldsymbol{p}_1=\boldsymbol{p}\\ \boldsymbol{p}_2=-\boldsymbol{p}}} \tag{37}$$

which becomes

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{1}{2c^2}\frac{m_1 - m_2}{(m_1 + m_2)^2}\frac{1}{\mu}\left[\left(\frac{p^2}{\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{p} + \frac{q_1 q_2}{r^3}\boldsymbol{r}(\boldsymbol{p}\cdot\boldsymbol{r})\right]. \tag{38}$$

This equation could also have been obtained by differentiating equation (34) with respect to time, while using that $\mathrm{d}\boldsymbol{K}/\mathrm{d}t = \boldsymbol{0}$.

To order $1/c^2$ we have that

$$\frac{1}{c^2}\dot{\boldsymbol{r}} = \frac{1}{c^2}\frac{\boldsymbol{p}}{\mu} \qquad \frac{1}{c^2}\dot{\boldsymbol{p}} = \frac{1}{c^2}\frac{q_1 q_2}{r^3}\boldsymbol{r} \tag{39}$$

where $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{p}}$ mean $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$, as usual. Hence, equation (38) is equivalent to the equation

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{1}{2c^2}\frac{m_1 - m_2}{(m_1+m_2)^2}\left[\frac{p^2}{\mu}\dot{\boldsymbol{r}} + \frac{q_1 q_2}{r}\dot{\boldsymbol{r}} + \dot{\boldsymbol{p}}(\dot{\boldsymbol{r}}\cdot\boldsymbol{r})\right]. \tag{40}$$

We now note that

$$\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\{(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{p}\} = \frac{1}{c^2}\left[(\boldsymbol{p}\cdot\boldsymbol{r})\dot{\boldsymbol{p}} + (\boldsymbol{p}\cdot\dot{\boldsymbol{r}})\boldsymbol{p} + (\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{p}\right]$$
$$= \frac{\mu}{c^2}\left(\dot{\boldsymbol{p}}(\dot{\boldsymbol{r}}\cdot\boldsymbol{r}) + \frac{p^2}{\mu}\dot{\boldsymbol{r}} + \frac{q_1 q_2}{r}\dot{\boldsymbol{r}}\right) \tag{41}$$

and this allows us to write

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{1}{2c^2}\frac{m_1 - m_2}{(m_1+m_2)^2}\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\{(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{p}\} \tag{42}$$

from which we get

$$\boldsymbol{R} = \boldsymbol{R}_0 + \frac{1}{2c^2}\frac{m_1 - m_2}{(m_1+m_2)^2}\frac{1}{\mu}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{p} \tag{43}$$

where $\boldsymbol{R}_0$ is an arbitrary vector. It defines the position of the centre of mass (the centroid) at a time when $(\boldsymbol{p}\cdot\boldsymbol{r})/c^2 = 0$. Each conic section possesses at least one such point, and expression (43) offers, therefore, a well-defined correlation between the orbital motion and the motion of the centre of mass for any electromagnetic two-body system. We need such a correlation in order to set up a well-defined expression for the constant of the motion defined by $\boldsymbol{K}$.

Inserting expression (43) into expression (34) for $\boldsymbol{K}$ finally gives

$$\boldsymbol{K} = \frac{1}{c^2}\mathcal{H}\boldsymbol{R}_0 + \frac{1}{2c^2}\frac{m_1 - m_2}{m_1+m_2}\frac{1}{\mu}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{p} - \frac{1}{2c^2}\frac{m_1 - m_2}{m_1+m_2}\left(\frac{p^2}{\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{r} \tag{44}$$

or

$$\boldsymbol{K} = \frac{1}{c^2}\mathcal{H}\boldsymbol{R}_0 - \frac{1}{2c^2}\frac{m_1 - m_2}{m_1+m_2}\boldsymbol{M} \tag{45}$$

where

$$\boldsymbol{M} = \left(\frac{p^2}{\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{r} - \frac{1}{\mu}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{p} = \frac{\boldsymbol{p}\times\boldsymbol{l}}{\mu} + \frac{q_1 q_2}{r}\boldsymbol{r}. \tag{46}$$

Here $\boldsymbol{M}$ is the Runge–Lenz vector defined by equation (1), with $\kappa = q_1 q_2$. We have thus reached our goal of demonstrating that the form of $\boldsymbol{M}$ is determined by the vector $\boldsymbol{K}$ in the centre-of-momentum system.

The fact that $\boldsymbol{K}$ is a constant of the motion implies that the Poisson bracket between $\mathcal{H}$ and $\boldsymbol{K}$ must vanish in the centre-of-momentum system, i.e.

$$[\mathcal{H}, \boldsymbol{K}] = \boldsymbol{0}. \tag{47}$$

With $\boldsymbol{K}$ given by expression (45) and $\mathcal{H}/c^2$ by expression (32), we find that

$$[\mathcal{H}, \boldsymbol{K}] = -\frac{1}{2c^2}\frac{m_1 - m_2}{m_1+m_2}[\mathcal{H}, \boldsymbol{M}] = -\frac{1}{2c^2}\frac{m_1 - m_2}{m_1+m_2}[H, \boldsymbol{M}] \tag{48}$$

where H is the non-relativistic Hamiltonian (33). Relation (47) obviously implies that

$$[H, \boldsymbol{M}] = \boldsymbol{0} \tag{49}$$

i.e. the Runge–Lenz vector must commute with the non-relativistic Hamiltonian (33).

The Poisson bracket relation (49) is identical with the Poisson-bracket relation (6). So, not only have we identified the physical origin of the Runge–Lenz vector, we have also shown that the Poisson-bracket relation (6) is a consequence of the Poisson-bracket relation (47). This relation reflects, in turn, the requirement that the dynamics must be in accordance with the structure of the inhomogeneous Lorentz group. This is a strong requirement on the form of a dynamics, as thoroughly discussed by Dirac [17]. Dirac refers, in fact, jointly to $\mathcal{H}$ and $\boldsymbol{K}$ as the Hamiltonians, thus stressing that the forms of $\mathcal{H}$ and $\boldsymbol{K}$ are strongly coupled by the group requirements. So strong is this coupling that its implications are also felt in the non-relativistic limit, namely through the requirement that $\boldsymbol{M}$ must exist as a constant of the motion in that limit.

4. The problem of two gravitating bodies

We shall now show that the Runge–Lenz vector emerges from the $\boldsymbol{K}$-vector of the gravitational two-body problem in a similar way as it does from the $\boldsymbol{K}$-vector of the electromagnetic two-body problem. The Lagrangian to order $1/c^2$ in this case is ([15, §81], [16, §106], [18]):

$$\mathcal{L} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{km_1m_2}{r} + \frac{1}{8c^2}\left(m_1v_1^4 + m_2v_2^4\right) - \frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2} + \frac{km_1m_2}{2c^2}\left(\frac{3(v_1^2+v_2^2)}{r} - \frac{7\boldsymbol{v}_1\cdot\boldsymbol{v}_2}{r} - \frac{(\boldsymbol{v}_1\cdot\boldsymbol{r})(\boldsymbol{v}_2\cdot\boldsymbol{r})}{r^3}\right). \tag{50}$$

The corresponding Hamiltonian is

$$\mathcal{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{km_1m_2}{r} - \frac{p_1^4}{8m_1^3c^2} - \frac{p_2^4}{8m_2^3c^2} + \frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2} - \frac{3k}{2c^2}\left(\frac{m_2}{m_1}\frac{p_1^2}{r} + \frac{m_1}{m_2}\frac{p_2^2}{r}\right) + \frac{k}{2c^2}\left(\frac{7\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{r} + \frac{(\boldsymbol{p}_1\cdot\boldsymbol{r})(\boldsymbol{p}_2\cdot\boldsymbol{r})}{r^3}\right) \tag{51}$$

and the $\boldsymbol{K}$-vector becomes

$$\boldsymbol{K} = -t\boldsymbol{P} + m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + \frac{1}{2c^2}\left(\frac{p_1^2}{m_1} - \frac{km_1m_2}{r}\right)\boldsymbol{r}_1 + \frac{1}{2c^2}\left(\frac{p_2^2}{m_2} - \frac{km_1m_2}{r}\right)\boldsymbol{r}_2. \tag{52}$$

Formally, the $\boldsymbol{K}$-vector results from the $\boldsymbol{K}$-vector of the electromagnetic problem by the substitution $q_1q_2 \to -km_1m_2$. The two Hamiltonians are, of course, not that simply related. Yet, it turns out that the equation of motion for the centre of mass in the centre-of-momentum system does result from equation (38) by the said substitution. Hence, the expression (43) for $\boldsymbol{R}$ is unchanged, and the expression for $\boldsymbol{K}$ in the centre-of-momentum system becomes similar to equation (45), the only difference being that the Runge–Lenz vector now has the form

$$\boldsymbol{M} = \frac{\boldsymbol{p}\times\boldsymbol{l}}{\mu} - \frac{km_1m_2}{r}\,\boldsymbol{r}. \tag{53}$$

Thus, the role played by the Runge–Lenz vector is the same whether one considers electromagnetic or gravitational interactions.

In connection with the expression (43) that we have derived for the centre of mass motion, it is worth while noticing that putting $\boldsymbol{K} = \mathbf{0}$ in (34) gives

$$(m_1 + m_2)\boldsymbol{R} = \frac{1}{2c^2}\frac{m_1 - m_2}{m_1 + m_2}\left(\frac{p^2}{\mu} + \frac{q_1 q_2}{r}\right)\boldsymbol{r}. \tag{54}$$

This expression for $\boldsymbol{R}$ does, of course, satisfy the equation of motion for $\boldsymbol{R}$, but the correlation with the orbital motion is not sufficiently precise to define an origin of coordinates independent of the state of motion. The expression was first derived by Fock ([15, §81], [19]), in his discussion of the gravitational two-body problem.

5. Discussion

With the above analysis, we have tied the Runge–Lenz vector to the generator, $\boldsymbol{K}$, of infinitesimal Lorentz transformations of a two-body system. The Poisson-bracket relation (6) also follows from the analysis.

As to the Poisson-bracket relations (4), they are reminiscent of the Poisson-bracket relations

$$[K_i, K_j] = -\frac{1}{c^2}\sum_k \epsilon_{ijk} L_k \tag{55}$$

which are imposed on the components of the vector $\boldsymbol{K}$ by the structure of the Lorentz group. They can, however, not be derived from these relations within our $1/c^2$ order description, since $\boldsymbol{M}$ appears in expression (45) as $\boldsymbol{M}/c^2$. The results of our analysis do, however, make relations (4) qualitatively understandable, when we invoke the following arguments.

We may write

$$[M_i, M_j] = \sum_k \epsilon_{ijk} A_k \tag{56}$$

where $\boldsymbol{A}$ is an axial vector. Our analysis has shown that $[M_i, H]$ must vanish, so $[A_i, H]$ must also vanish. $\boldsymbol{A}$ is therefore a constant of the motion in the non-relativistic dynamics. This allows us to conclude [20, 21], that $\boldsymbol{A}$ may be written on the form

$$\boldsymbol{A} = \sigma \boldsymbol{l} \tag{57}$$

where σ is a constant of the motion. Its dimension must be energy divided by mass. The simple choice

$$\sigma = -\frac{2H}{\mu} \tag{58}$$

reproduces relation (4).

This concludes our discussion of the Runge–Lenz vector and its connection with the generator of infinitesimal Lorents transformations. We emphasize that the discussion has been entirely classical. The Runge–Lenz vector is, however, also a constant of the motion in non-relativistic, spin-free quantum mechanics. As mentioned in the introduction, this follows by substituting commutators for Poisson brackets according to equation (7). However, the quantum-mechanical Kepler problem is incomplete without the introduction of the spin. With spin included, the proper point of departure is the Dirac equation and the so-called Johnson–Lippmann operator [22] which accounts for the dynamical symmetry in the Dirac–Kepler problem. An analysis along similar lines as those followed in the present paper—including the crucial transition to the two-body problem—would be extremely difficult. It is, however, worthwhile noting that the non-relativistic limit of the Johnson–Lippmann operator equals $-\boldsymbol{\sigma} \cdot \boldsymbol{M}$ where $\boldsymbol{\sigma}$ is the Pauli spin vector and $\boldsymbol{M}$ is the Runge–Lenz vector [23, 24].

Acknowledgments

It is a pleasure to thank Professor R F O'Connell for stimulating discussions. This work was supported by the Carlsberg Foundation and the Danish Natural Science Research Council.

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